Chapter 11

Mapping II

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11.1 Introduction

An aperture synthesis array measures the visibilities at discrete points in the uv-domain. The visibilities are Fourier transformed to get the *Dirty Map* and the weighted uv-sampling function is Fourier transformed to get the *Dirty Beam* using the efficient FFT algorithm. This lecture describes the entire chain of data processing required to inverted the visibilities recorded as a function of (u, v, w), and the resulting errors/distortions in the final image. In this entire lecture, the '$\star$' operator represents convolution operation, the '.' operator represents point-by-point multiplication and the '$\rightleftharpoons$' operator represents the Fourier transform operator.

As described earlier, the visibility V measured by an aperture synthesis telescope is related to the sky brightness distribution I as

$$V \rightleftharpoons I, \tag{11.1.1}$$

where $\rightleftharpoons$ denotes the Fourier Transform. The above equation is true only for the case of continuous sampling of the uv-plane such that V is measured for all values of (u, v). However since there are finite antennas in an array, uv-plane is sampled at descreet uv points and Eq. 11.1.1 has to be written as

$$V.S \rightleftharpoons I * DB (= I^d), \tag{11.1.2}$$

where I^d is the *Dirty Map*, I is the true brightness distribution, DB is the *Dirty Beam* and S is the uv-sampling function given by

$$S(u, v) = \sum_k \delta(u - u_k, v - v_k), \tag{11.1.3}$$

where u_k and v_k are the actual (u, v) points measured by the telescope. The pattern of all the measured (u, v) points is referred to as the uv-coverage.

This function essentially assigns a weight of unity to all measured points and zero everywhere else in the uv-plane. Fourier transform of S is referred to as the *Dirty Beam*. As written in Eq. 11.1.2, the *Dirty Beam* is the transfer function of the instrument used as an imaging device. The shape of the *Dirty Beam* is a function of the uv-coverage which in turns is a function of the location of the antennas. *Dirty Beam* for a fully covered uv-plane will be equal to $sin(\pi l\lambda/u_{max})/(\pi l\lambda/u_{max})$ where u_{max} is the largest antenna spacing for which a measurement is available. The width of the main lobe of this function is proportional to λ/u_{max}. The resolution of such a telescope is therefore roughly λ/u_{max} and u_{max} can be interpreted as the size of an equivalent lens. For a real uv-coverage however, S is not flat till $u_{\max}$ and has 'holes' in between representing un-sampled (u, v) points. The effect of this missing data is to increase the side-lobes and make the *Dirty Beam* noisy, but in a deterministic manner. Typically, an elliptical gaussian can be fitted to the main lobe of the *Dirty Beam* and is used as the resolution element of the telescope. The fitted gaussian is referred to as the *Synthesized Beam*.

The *Dirty Map* is a convolution of the true brightness distribution and the *Dirty Beam*. I^d is almost never a satisfactory final product since the side-lobes of DB (which are due to missing spacings in the uv-domain) from a strong source in the map will contaminate the entire map at levels higher than the thermal noise in the map. Without removing the effect of DB from the map, the effective RMS noise in the map will be much higher than the thermal noise of the telescope and will result into obscuration of faint sources in the map. This will be then equivalent to reduction in the dynamic range of the map. The process of De-convolving is discussed in a later lecture, which effectively attempts to estimate I from I^d such that $(I - I^d) * DB$ is minimized consistent with the estimated noise in the map.

To use the FFT algorithm for Fourier transforming, the irregularly sampled visibility $V(u, v)$ needs to be gridded onto a regular grid of cells. This operation requires interpolation to the grid points and then re-sampling the interpolated function. To get better control on the shape of the *Dirty Beam* and on the signal-to-noise ratio in the map, the visibility is first re-weighted before being gridded. These operations are described below.

11.2 Weighting, Tapering and Beam Shaping

The shape of the *Dirty Beam* can be controlled by multiplying S with other weighting functions. Note that the measured visibilities already carry a weight which is a measure of the signal-to-noise ratio of each measurement. Since there is no control on this weight while mapping, it is not explicitly written in any of equations here but is implicitly used.

Full weighting function W as used in practice is given by

$$W(u, v) = \sum_k T_k D_k \delta(u - u_k, v - v_k). \tag{11.2.4}$$

The function T_k is the 'uv-tapering' function to control the shape of DB and D_k is the 'density-weighting' function used in all imaging programs. If S was a smooth function, going smoothly to zero beyond the maximum sampled uv-point, DB would also be smooth with no side lobes (e.g. if S was a gaussian). However, S is collection of delta functions with gaps in between (for the missing uv-points not measured by the telescope) and has a sharp cut-off at the limit of uv-coverage. This results into DB being a highly non-smooth function with potentially large side-lobes.

As is evident from the plots of uv-coverage, the density of uv-tracks decreases away from the origin. If one were to use the local average of the uv-points in the uv-plane for mapping as is done in the gridding operation described below, the signal-to-noise ratio of the points would be proportional to the number of uv-points averaged. Since the density of measured uv-points is higher for smaller values of u and v, visibilities for shorter spacings get higher weightage in the visibility data effectively making the array relatively more sensitive to the broader features in the sky. The function D_k controls the weights resulting from non-uniform density of the points in the uv-plane.

Both T_k and D_k provide some control over the shape of the *Dirty Beam*. T_k is used to weight down the outer edge of the uv-coverage to decrease the side-lobes of DB at the expense decreasing the spatial resolution. D_k is used to counter the preferential weight that the uv-points get closer to the origin at the expense of degrading the signal-to-noise ratio.

T_k is a smoothly varying function of (u, v) and is often used as $T(u_k, v_k) = T(u_k)T(v_k)$. For most imaging applications, $T(u_k, v_k)$ is a circularly symmetric gaussian. However other forms are also occasionally used.

Two forms of D_k are commonly used. When $D_k = 1$ for all values of (u, v), it is referred to as 'natural weighting' were the natural weighting of the uv-coverage is used as it is. This gives best signal-to-nose ratio and is good when imaging weak compact sources but is undesirable for extended sources where both large scale and small scale features are present.

When $D_k = 1/N_k$ where N_k is a measure of the local density of uv-points around (u_k, v_k), it is referred to as 'uniform weighting' where an attempt is made to assign uniform weights to the entire covered uv-plane. In standard data reduction packages available for use currently (*AIPS, SDE, Miriad*), while re-gridding the visibilities (discussed below), N_k is equal the number of uv-points within a given cell in the uv-plane. However it can be shown that this can result into serious errors, referred to as *catastrophic gridding error* in some pathological cases. This problem can be handled to some extend by using better ways of estimating the local density of uv-points (Briggs, 1995).

Eq. 11.1.2, using the weighted sampling function W is written as

$$(V.S.W) \rightleftharpoons (I * DB). \tag{11.2.5}$$

Note that $DB \rightleftharpoons S.W$, i.e. the *Dirty Beam* is the Fourier transform of the weighted sampling function.

11.3 Gridding and Interpolation

The inversion of the visibilities to make the *Dirty Map* is done using FFT algorithm which requires that the function be sampled at regular intervals and the number of samples be power of 2. For the case of mapping the sky using an aperture synthesis telescope, this implies that the visibility data be available on a regular 2D grid in the uv plane. Thus re-gridding of the data onto a regular grid is required by potentially interpolating the visibility to the grid points, since the visibility function $V(u, v)$ is measured at discrete points (u, v) which are not assured to be at regular intervals along the u and v axis.

Interpolation of V is done by multiplying a function and averaging all the measured points which lie within the range of the function with a finite support base, centered at each grid point. The resultant average value is assigned to the corresponding grid point. This operation is equivalent to discrete convolution of V with the above mentioned function and then sampling this convolution at the grid points. The convolving function is referred to as the Gridding Convolution Function (GCF). There are other ways of doing this interpolation. However the interpolation in practice is done by convolution since this results into predictable results in the map plane which are easy to visualize. Also using GCF with finite support base results into each grid point getting the value of the local average of the visibilities.

After gridding Eq. 11.2.5 becomes

$$(V.S.W) * C \rightleftharpoons (I * DB).c, \tag{11.3.6}$$

where C represents the GCF and $c \rightleftharpoons C$.

The effect of gridding the visibilities on the map is to multiply the map with function c and since C has a finite support base (i.e. is of finite extent), c is infinite in extent which result into aliasing in the map plane (the other cause of aliasing could be under-sampling of the uv-plane). The amplitude of the aliased component from a position (l, m) in the map is determined by $c(l, m)$. Ideally therefore, this function should be rectangular function with the width equal to the size of the map and smoothly going to zero immediately outside the map. However from the point of efficiency of the gridding process, this is not possible, and GCF used in practice have a trade-off between the roll-off properties at the edge and flatness within the map.

Since the *Dirty Map* is multiplied by c, if c is well known, then effect of convolution by the GCF can be removed by point-wise division of *Dirty Map* and *Dirty Beam* given by $\bar{I}^d = I^d/c$ and $\bar{DB} = DB/c$ for later processing, particularly in deconvolution of I^d. In practice however, this division is not carried out by evaluating $c(l, m)$ over the map. Instead, for efficiency purposes, this function is kept in the computer memory tabulated with a resolution typically $1/100$ times the size of the cell in the image.

To take the Fourier transform of $(V.S.W) * C$ using the FFT algorithm one needs to sample the right hand side of Eq. 11.3.6 by multiplication with the re-sampling function R given by

$$R(u, v) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(j - u/\Delta u, k - v/\Delta v), \tag{11.3.7}$$

where Δu and Δv are the cell size in the uv-domain. Eq. 11.3.6 then becomes

$$R.((V.S.W) * C) \rightleftharpoons r * ((I * DB).c), \tag{11.3.8}$$

where $R \rightleftharpoons r$. Right hand side of this equation then is the approximation of I^d obtained in practice. As discussed in earlier lecture, FFT generates a periodic function (due to the presence of R in the left hand side of Eq. 11.3.8) and I^d represents one period of such a function. To map an angular region of sky of size $(N_l\Delta l, N_m\Delta m)$, using the Nyquist sampling theorem we get $N_l\Delta u = 1/\Delta l$ and $N_m\Delta v = 1/\Delta m$ where Δl and Δm is the cell size in the map and Δu and Δv are cell sizes in the uv-plane.

C is usually real and even and is assumed to be separable as $C(u, v) = C_1(u)C_2(v)$. Various GCFs used in practice are listed below. Functions listed below are in 1-dimension and are truncated (set to zero) for $|u| \geq m\Delta u/2$ where Δu is the size of the grid and m is the number of such cells used.

1. 'Pillbox' function

$$C(u) = \left\{ \begin{array}{c} 1, \ |u| < m\Delta u/2 \\ 0, \ otherwise \end{array} \right\}. \tag{11.3.9}$$

   This amounts to simple averaging of all the uv-points with in the rectangle defined by Eq. 11.3.9. However since its Fourier transform is $sinc$ with large side lobes, it provides poor alias rejection and is almost never used but is useful for intuitive understanding.

2. Truncated exponential function

$$C(u) = e^{\frac{-|u|^{\alpha}}{w\Delta u}}. \tag{11.3.10}$$

Typically $m = 6$, $w = 1$ and $\alpha = 2$ is used and c can be expressed in terms of error function.

3. Truncated $sinc$ function

$$C(u) = sinc\left(\frac{u}{w\Delta u}\right). \tag{11.3.11}$$

For $m = 6$ and $w = 1$, this is the normal $sinc$ function expressed in terms of sin function. As m increases, the Fourier transform of this function approaches a step function which is constant over the map and zero outside.

4. Sinc exponential function

$$C(u) = e^{\frac{-|u|^{\alpha}}{w_1\Delta u}} sinc\left(\frac{u}{w_2\Delta u}\right). \tag{11.3.12}$$

For $m = 6$, $w_1 = 2.52$, $w_2 = 1.55$, $\alpha = 2$, the above equation reduces to multiplication of gaussian with the exponential function. This optimizes between the flat response of exponential within the map and suppression of the side-lobes due the presence of the gaussian.

5. Truncated spheroidal function

$$C(u) = |1 - \eta^2(u)|^{\alpha}\phi_{\alpha 0}(\pi m/2.\eta(u)), \tag{11.3.13}$$

where $\phi_{\alpha 0}$ is the 0-order spheroidal function, $\eta(u) = 2u/m\Delta u$ and $\alpha > -1$.

Of all the square integrable functions, this is the most optimal in the sense that it has maximum contribution to the normalized area from the part of $c(l)$ which is with in the map. This is referred to as the *energy concentration ratio* expressed as $\frac{\int_{map} |c(l)|^2 dl}{\int_{-\infty}^{\infty} |c(l)|^2 dl}$ is maximized.

11.4 Bandwidth Smearing

The effect of a finite bandwidth of observation as seen by the multiplier in the correlator, is to reduce the amplitude of the visibility by a factor given by $sin(\pi l\Delta\nu/\nu_o\theta)/(\pi l\Delta\nu/\nu_o\theta)$, where θ is angular size of the synthesized beam, ν_o is the center of the observing band, l is location of the point source relative to the field center and $\Delta\nu$ is the bandwidth of the signal being correlated.

The distortion in the map due to the finite bandwidth of observation can be visualized as follows. For continuum observations, the visibility data integrated over the bandwidth $\Delta\nu$ is treated as if the observations were made at a single frequency ν_o, the central frequency of the band. As a result the u and v co-ordinates and the value of visibilities are correct only for ν_o. The true co-ordinate at other frequencies in the band are related to the recorded co-ordinates as

$$(u, v) = \left(\frac{\nu_o u_\nu}{\nu}, \frac{\nu_o v_\nu}{\nu}\right). \tag{11.4.14}$$

Since the total weights W used while mapping does not depend on the frequency, the relation between the brightness distribution and visibility at a frequency ν becomes

$$V(u,v) = V\left(\frac{\nu_o u_\nu}{\nu}, \frac{\nu_o v_\nu}{\nu}\right) \rightleftharpoons \left(\frac{\nu}{\nu_o}\right)^2 I\left(\frac{l\nu}{\nu_0}, \frac{m\nu}{\nu_0}\right). \qquad (11.4.15)$$

Hence the contribution of $V(u,v)$ to the brightness distribution get scaled by $(\nu/\nu_o)^2$ and the co-ordinates gets scaled by (ν/ν_o). The effect of the scaling of the co-ordinates, assuming a delta function for the *Dirty Beam*, is to smear a point source at position (l,m) into a line of length $(\Delta\nu/\nu_o)\sqrt{l^2+m^2}$ in the radial direction. This will get convolved with the *Dirty Beam* and the total effect can be found by integrating the brightness distribution over the bandwidth as given in Eq. 11.4.15

$$I^d(l,m) = \left[\frac{\int_0^\infty |H_{RF}(\nu)|^2 \left(\frac{\nu}{\nu_o}\right)^2 I\left(\frac{l\nu}{\nu_0}, \frac{m\nu}{\nu_0}\right) d\nu}{\int_0^\infty |H_{RF}(\nu)|^2 d\nu}\right] * DB_o(l,m), \qquad (11.4.16)$$

where $H_{RF}(\nu)$ is the band-shape function of the RF band and DB_o is the *Dirty Beam* at frequency ν_o. If one represents the synthesized beam as a gaussian function of standard deviation $\sigma_b = \theta_b/\sqrt{8ln2}$ and the bandpass represented by a rectangular function of width $\Delta\nu$, the fractional reduction in the strength of a source located at a radial distance $r = \sqrt{l^2+m^2}$ is given by

$$R_b = 1.064\frac{\theta_b \nu_o}{r\Delta\nu} erf\left(0.833\frac{r\Delta\nu}{\theta_b\nu_o}\right). \qquad (11.4.17)$$

Eq. 11.4.16 is equivalent to averaging large number of maps made from monochromatic visibilities at ν. Since each of such maps would scale by a different factor, the source away from the center would move along the radial line from one map to another, producing the radial smearing convolved with the *Dirty Beam*. Since the source away from the center is elongated radially, its side-lobes (because of the *Dirty Beam*) will also be elongated in the radial direction. As a result the side-lobes of distant sources will be elongated at the origin but not towards 90^o angle from the vector joining the source and the origin.

The effect of bandwidth smearing can be reduced if the RF band is split into frequency channels with smaller channel widths. This effectively reduces the $\Delta\nu$ as seen by the mapping procedure and while gridding the visibilities then, the u and v can be computed separately for each channel and assigned to the correct uv-cell. The FX correlator used in GMRT provides up to 128 frequency channels over the bandwidth of observation.

11.5 Time Average Smearing

As discussed before, the u and v co-ordinates of an antenna are a function of time and continuously change as earth rotates generating the uv-coverage. To improve the signal-to-noise ratio as well as reduce the data volume, the visibility function $V(u,v)$ is recorded after finite integration in time (typically 10-20s for imaging projects) and the average value of the real and imaginary parts of V are used for average values of u and v over the integration time. Effectively then, the assigned values of u and v for each visibility point is evaluated for a time which is wrong from the correct (instantaneous) time by a maximum of $\tau/2$ where τ is the integration time.

In the map domain, the resulting effect can be visualized by treating the resulting map from the time average visibilities as the average for a number of maps made from the instantaneous (un-averaged) visibilities. The baseline vectors in the uv-domain follow the loci of the uv-tracks (which are parabolic tracks) and rotate at an angular velocity equal to the that of earth, ω_e. Since a rotation of one domain results into a rotation by an equal amount in the conjugate domain in a Fourier transform relation, the effect in the map domain is that the instantaneous maps also are rotated with respect to each other, at the rate of ω_e. Hence, a point source located at (l, m) away from the center of the map would get smeared in the azimuthal direction. This effect is same as the smearing effect due to finite bandwidth of observations, but in an orthogonal direction.

11.6 Zero-spacing Problem

Since visibility and the brightness distribution are related via a Fourier transform, $V(0,0)$ measures the total flux from the sky. However, since the difference between the antenna positions is always finite, $V(0,0)$ is never measured by an interferometer. For a point source, it is easy to estimate this value by extrapolation from the smallest u and v for which a measurement exist, since V as a function of baseline length is constant. However for an extended source, this value remains unknown and extrapolation is difficult.

For the purpose of understanding the effect of missing zero-spacings, we can multiply the visibility in Eq. 11.3.6 by a rectangular function which is 0 around $(u, v) = (0, 0)$ and 1 elsewhere. In the map domain then, the *Dirty Map* gets convolved with the Fourier transform of this function, which has a central negative lobe. As a result, extended sources will appear to be surrounded by negative brightness in the map which cannot be removed by any processing. This can only be removed by either estimating the zero-spacing flux while restoring I from I^d or V, or by supplying the zero-spacing flux as an external input to the mapping/deconvolution programs. The Maximum Entropy class of image restoration algorithms attempt to estimate the zero-spacing flux, while the CLEAN class of image restoration algorithms needs to be supplied this number externally. Both these will be discussed in the later lectures.

11.7 Further Reading

1. Interferometry and Synthesis in Radio Astronomy;Thompson, A.Richard, Moran, James M., Swenson Jr., George W.; Wiley-Interscience Publication, 1986.

2. Synthesis Imaging In Radio Astronomy; Eds. Perley, Richard A.,Schwab, Frederic R., and Bridle, Alan H.; ASP Conference Series, Vol 6.

3. High Fidelity Deconvolution of Moderately Resolved Sources; Briggs Daniel; Ph.D. Thesis, 1995, The New Maxico Institute of Mining and Technology, Socorro, New Mexico, USA.