Polarisation 4

- Images are made by Fourier transforming visbilities/correlations as a function of projected baselines to obtain a map as a function of sky direction
- For full polarisation observations, the correlation between antennas a and b is a 2x2 complex matrix and its Fourier transform an image of the Stokes parameters in the sky

$$\mathbf{V}_{ab} = \left\langle \begin{bmatrix} E_{a,x} \\ E_{a,y} \end{bmatrix} \begin{bmatrix} E^*_{b,x} & E_{b,y} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} E_{a,x} E^*_{b,x} & E_{a,x} E^*_{b,y} \\ E_{a,y} E^*_{b,x} & E_{a,y} E^*_{b,y} \end{bmatrix} \right\rangle = \begin{bmatrix} I+Q & U-iV \\ U+iV & I-Q \end{bmatrix}$$

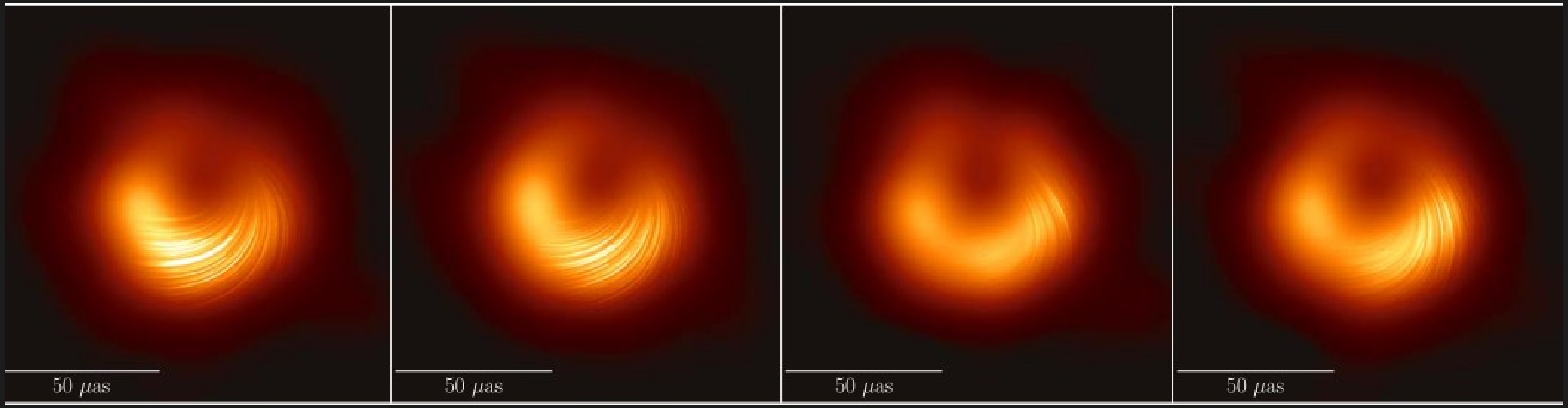
Calibration and deconvolution

- In practice, the measured quantities differ from the ideal because of instrumental and propagation effects, and these can be calibrated by observations of point sources, including polarised ones
- After applying calibration, the visibilities still do not cover the u-v plane, so the ‘point spread function’ has undesirable features – the map is ‘cleaned’
- Deconvolution is a euphemism for creating data where you have no measurements - based on prejudice :-)
- This process applied alone, gave dynamic ranges of ~100 – early VLA specs, Ryle Nobel lecture quite pessimistic on future progress at shorter wavelengths

The self calibration revolution

- Necessity the mother of invention – MERLIN was phase unstable and VLBI phases are difficult to pin down
- Cornwell / Wilkinson and Schwab invented self cal 1981
- Since deconvolution anyway ‘solves’ for missing visibilities, why not let it ‘solve’ for the unknown antenna based gains as well, which are much fewer in number?
- When selfcal is good it is very very good, dynamic ranges went up by two orders of magnitude.
- But when data is sparse, it doesn't work so well

Fast forward to the EHT



Polarised images of M87 at 230 GHz   field is 120 microarcseconds. Extensive efforts to check for internal consistency and assess uncertainties, Multiple and independent teams made images by different methods

Counting all closure quantities

- Snapshot visibilities have $N(N-1)$ real parameters
- N complex gains have $2N-1$ real parameters. The -1 is for the overall phase which doesn't affect visibilities
- We have $N(N-1) + 2N-1$ unknowns but only $N(N-1)$ measurements
- We can eliminate gains and get a smaller number of identities between measured and true visibilities
- e.g. $x+y+z=1$; $x-3y+2z=3$; imply $-x-5y=1$

Closure quantities, copolar case

From the earliest days of radio astronomy, ways were found to cancel antenna dependent complex gains, in the copolar/ scalar case

$$e_a^M = g_a e_a^S \text{ so } v_{ab}^M = g_a \langle e_a^S e_b^{*S} \rangle g_b^* = g_a v_{ab}^S g_b^*$$

Closure phase (Jennison 1958)

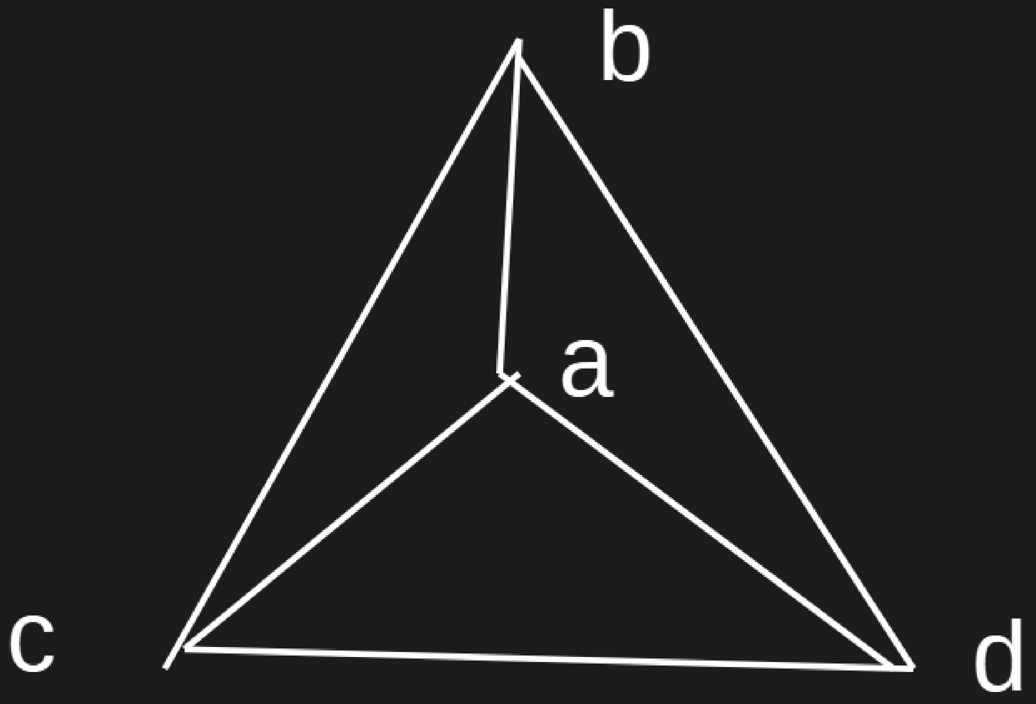
$$v_{ab}^M v_{bc}^M v_{ca}^M = g_a g_a^* g_b g_b^* g_c g_c^* v_{ab}^S v_{bc}^S v_{ca}^S$$

Closure amplitude (Twiss, Carter,Little 1960)

$$\frac{v_{ab}^M v_{cd}^M}{v_{ad}^M v_{cb}^M} = \frac{v_{ab}^S v_{cd}^S}{v_{ad}^S v_{cb}^S}$$

Counting closure phases

- Triangles grow as N^3 and quads as N^4 while visibilities have only $N(N-1)$ real parameters so clearly we need to find independent sets
- Basing triangles at one vertex does the job we can produce *bcd* from *abc*,*acd* and *adb* so $(N-1)(N-2)/2$ independent quantities



Counting closure amplitudes

- Eliminating the *g*'s will give $N^2 - N - 2N + 1 = N^2 - 3N + 1$ real combinations of true visibilities in terms of measured ones.
- That leaves $(N^2 - 3N + 1) - (N-1)(N-2)/2 = N(N-3)/2$ closure amplitudes
- A direct construction of independent closure amplitudes not given in TMS, an elegant one came from the EHT group (Blackburn et al 2019)

A more geometric view

- The space of measured visibilities has *$N(N-1)$* real dimensions and the true visibilities are a point in this space
- This point gets spread out into an 'orbit' when acted upon by *$2N-1$* variable gain parameters, so the measured visibilities lie on *$2N-1$* dimensional surfaces – a ' foliation'
- The 'co-dimension' of each surface is *$N(N-1)-(2N-1)= N^2-3N+1$* meaning it takes that many co-ordinates to specify which surface we are on
- These co-ordinates are constant on each surface, so that is the number of invariants independent of gains

Imaging with closure quantities

- Early efforts in VLBI, comparing models to the closure quantities they predict – so ‘forward modelling’
- When selfcal converges then no need for closure quantities – the solution fits the data to within the noise with the ‘solved’ gains applied and hence satisfies all closure quantities
- Closure is useful in situations where selfcal itself is shaky
- Straightforward imaging tools don't quite work so alternatives like imaging purely with closure quantities explored by the EHT group (Chael et. Al 2018)

Closure with polarisation

(work with N.Thyagarajan, J.Samuel, Vinay Kumar)

- The true visibility is sandwiched between two gain matrices to give the measured visibility

$$\boldsymbol{V}_{ab}^{M} = \langle \boldsymbol{E}_a^M \boldsymbol{E}_b^{M+} \rangle = \boldsymbol{G}_a \langle \boldsymbol{E}_a^S \boldsymbol{E}_b^{S+} \rangle \boldsymbol{G}_b^{+} = \boldsymbol{G}_a \boldsymbol{V}_{ab}^{S} \boldsymbol{G}_b^{+}$$

- To cancel the gains, one has to put on a hat

Define $\hat{\boldsymbol{A}} = \left(\boldsymbol{A}^{+}\right)^{-1}$ for any matrix $\boldsymbol{A}$

- Hat does not reverse the order of a product

When we form $\boldsymbol{V}_{ab} \hat{V}_{bc} V_{cd}$ the intermediate $\boldsymbol{G}_b$ and $\boldsymbol{G}_c$ cancel

Products around loops

A matrix product around an even numbered loop with alternate hats picks up a ***G_a*** on the left and ***G_a^{-1}*** on the right

$$V^M_{ab}\hat{V}^M_{bc}V^M_{cd}\hat{V}^M_{da} = G_a V_{ab}\hat{V}_{bc}V_{cd}\hat{V}_{da}G_a^{-1}$$

Determinants and traces of even numbered loops therefore cancel all matrix valued gains and are true ‘invariants’

-**Broderick and Pesce 2000**

Not clear even after this work that one had a complete set of invariants

More numerology

- The total number of real quantities entering the visibilities is *$8N(N-1)/2 = 4N(N-1)$*
- The number of parameters in the gains is *$8N-1$*
- We expect *$4N^2 -12N +1$* real invariants but counting alone doesn't help to construct invariants
- There is a numerical strategy to count independent invariants, based on the geometric picture. Perturb the visibilities and see how the given set of invariants varies.

The task is to construct the right number of invariants

- The breakthrough came from J.Samuel who used triangles

$$V_{ab}^{M}\hat{V}_{bc}^{M}V_{ca}^{M}=G_{a}V_{ab}\hat{V}_{bc}V_{ca}G_{a}^{+}$$

- The action of $G_a\, M\, G_a^+$ is related to Lorentz transformations plus scaling of two four vectors constructed from M
- The connection between Lorentz transformations and complex matrices with determinant 1 goes back to the early days of relativity
- However, the result we need is easily verified

The Lorentz connection

$$\boldsymbol{M} = \begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix}; \quad det(M) = t^2 - z^2 - x^2 - y^2$$

Any 2x2 complex matrix ***M*** can be written in the above form, with complex *t,x,y,z* Multiplying by other matrices with determinant 1 will preserve *det (M)* and will hence act like a Lorentz transformation on *t,x,y,z* . **(Britton 2000)**

Further, if we have ***G*** on the left and ***G***$^{+}$ on the right then this transformation is real – it doesn't mix real and imaginary parts of *t,x,y,z* *det (* ***GG***$^{+}$ *)* acts like an overall scale factor Therefore, every such matrix ***M*** gives us two 4-vectors for free

The complete set of invariants

- Construct all triangles starting at a given vertex
- Construct the matrices *M* for all of these, $A=(N-1)(N-2)/2$ in number
- Use the first two triangles to get four 4 vectors and their ten dot products
- Use this basis to construct 8 more invariants for each later triangle.
- We get $8A-7= 4N^2-12N+1$ real invariant quantities
- Scale is taken care of by normalising

More...

- This ‘triangles’ approach clarifies even the co-polar case – no Lorentz transformations needed, just scale. Closure amplitudes and closure phases are unified
- The invariants are highly non-linear combinations of Fourier coefficients. No clear intepretation in the image space. So simulations needed, even AI?
- Need to look at noise properties – what if one of the hatted matrices is close to singular?
- Incomplete graph, Single polarisation antenna.
- Real data

Topics for 11 Dec talk

- Representation of absorbing / lossy systems on the Poincare sphere.
- Connection with Lorentz transformations on the celestial sphere
- The formal analogy of polarised light with spin half systems
- Polarisation over the whole sky: Scalar feeds, Huygens sources.
- The three dimensional space of polarisation and phase