Polarisation 1

Rajaram Nityananda
ICTS-TIFR
Bengaluru 560089

Newton on polarisation

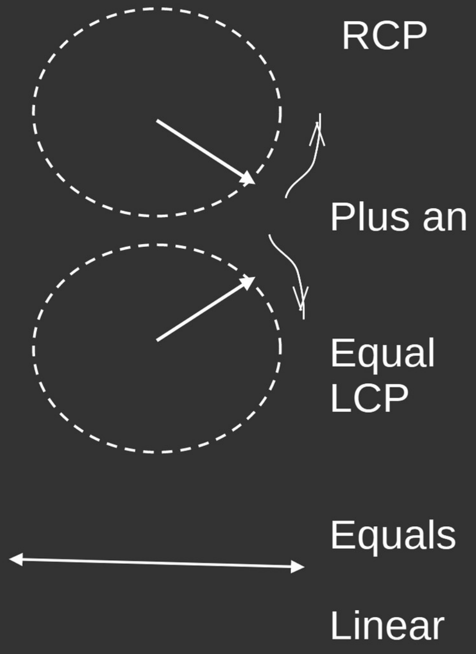
scienceresource.com/archive

Every Ray of Light has therefore two opposite Sides, originally endued with a Property on which the[Pg 361] unusual Refraction depends, and the other two opposite Sides not endued with that Property.

(Newton, in Opticks, regarding double refraction)

Some history

- Linear polarisation inferred from double refraction in calcite - took a hundred plus years, from Huygens to Malus of the $cos^2\theta$ law fame
- Circular polarisation helped Fresnel to understand the rotation of linear polarisation in quartz (Arago, Herschel), solutions (Biot, Pasteur), and a material in a magnetic field (Faraday).



Algebra of polarisation

- Single dish, ideal crossed dipoles at focus in *x*-*y* plane
- Single frequency $\omega = 2\pi\nu$, signal received on axis, with different amplitudes and phases along *x* and *y*

$$E_x = a_1 \cos(\omega t - \phi_1) \;=\; \Re\left(a_1 \exp(i\phi_1)\exp(-i\omega t)\right)$$

$$E_y = a_2 \cos(\omega t - \phi_2) \;=\; \Re\left(a_2 \exp(i\phi_2)\exp(-i\omega t)\right)$$

- One can eliminate time to get an ellipse

Complex simplicity

- Suppress the time dependence. Pack into a complex column vector in two dimensions – a 2x1 matrix

$$\boldsymbol{E} = \begin{bmatrix} a_1 \exp(i\phi_1) \\ a_2 \exp(i\phi_2) \end{bmatrix} \equiv \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Can add these vectors (coherent superposition) multiply overall by a complex number, (amplification/ attenuation) ,

Can change basis – not just rotation in the x-y plane

The circular basis

Instead of $z_1\begin{bmatrix}1\\0\end{bmatrix}+z_2\begin{bmatrix}0\\1\end{bmatrix}$ we write $\frac{z_R}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}+\frac{z_L}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$

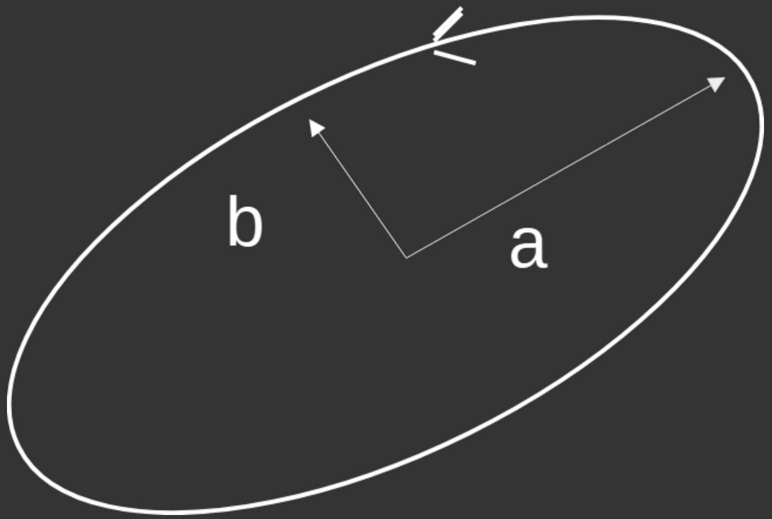
This leads to $\begin{bmatrix}z_1\\z_2\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\i & -i\end{bmatrix}\begin{bmatrix}z_R\\z_L\end{bmatrix}$

Inverting, we get $\begin{bmatrix}z_R\\z_L\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\1 & i\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix}$

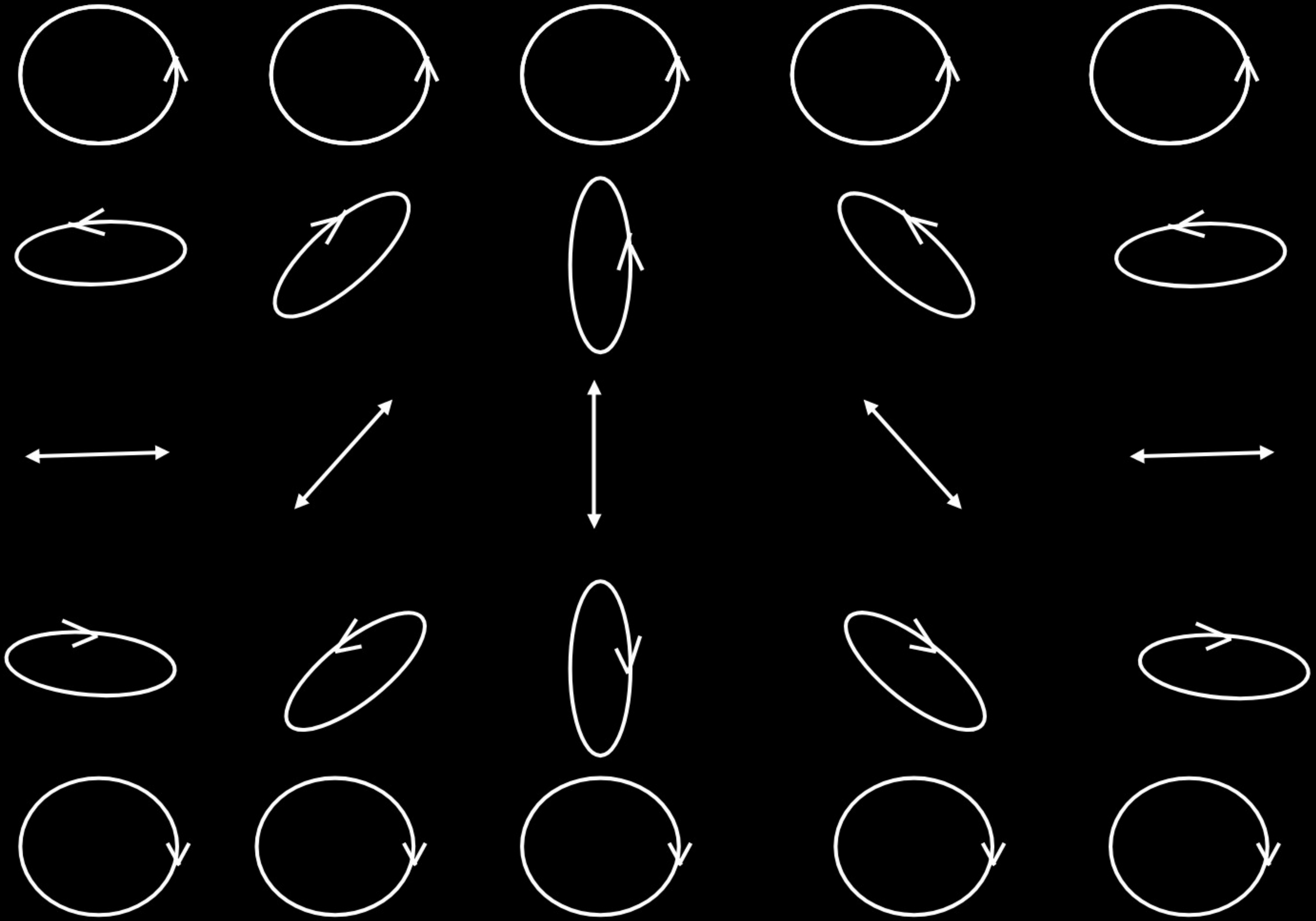
Geometry of polarisation

- Early work on polarisation focused on the path taken by a particle of the ‘ether’ - nowadays, path of electric field vector in the transverse plane
- Most general case, is an ellipse, traversed clockwise (right rotating) or counterclockwise (left rotating) in the *x-y* plane as viewed toward the source.
- Concentrate on the shape of this curve, not its size, and not where we happen to be on it which anyway changes rapidly. ‘state’ means ignore intensity, phase

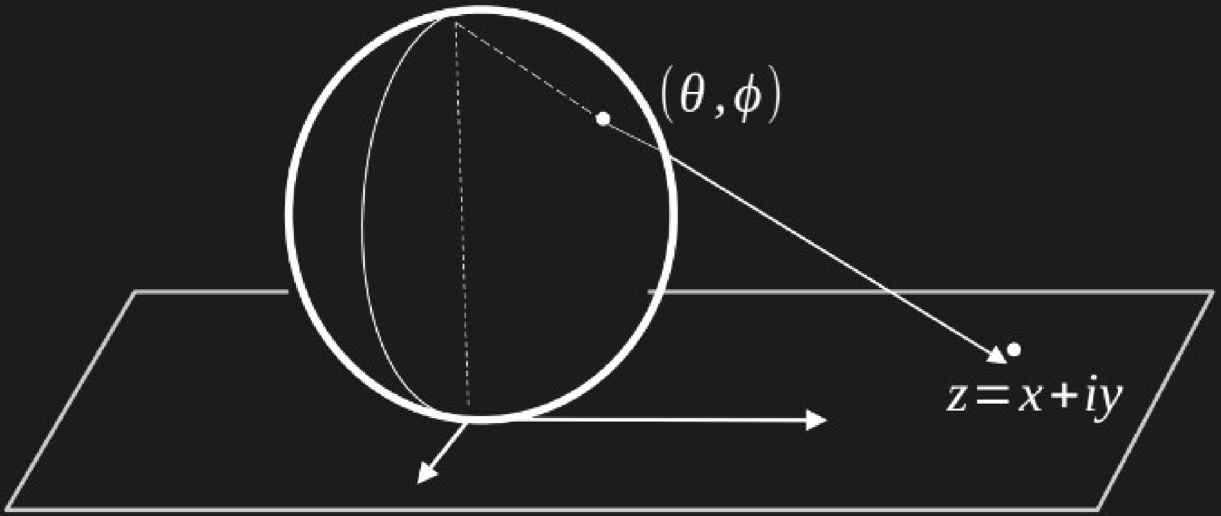
All possible states of polarisation



- Major axis angle ***α*** with x-axis ranges from *0* to *π*
- *b* ranges from – a to +a ***β=arctan (b/a)*** ranges from -π/4 to +π/4. Sign of ***b*** gives sense
- ***2 α*** and ***2 β*** are like RA and dec, or longitude and latitude on a sphere

Poincare did it by stereographic projection



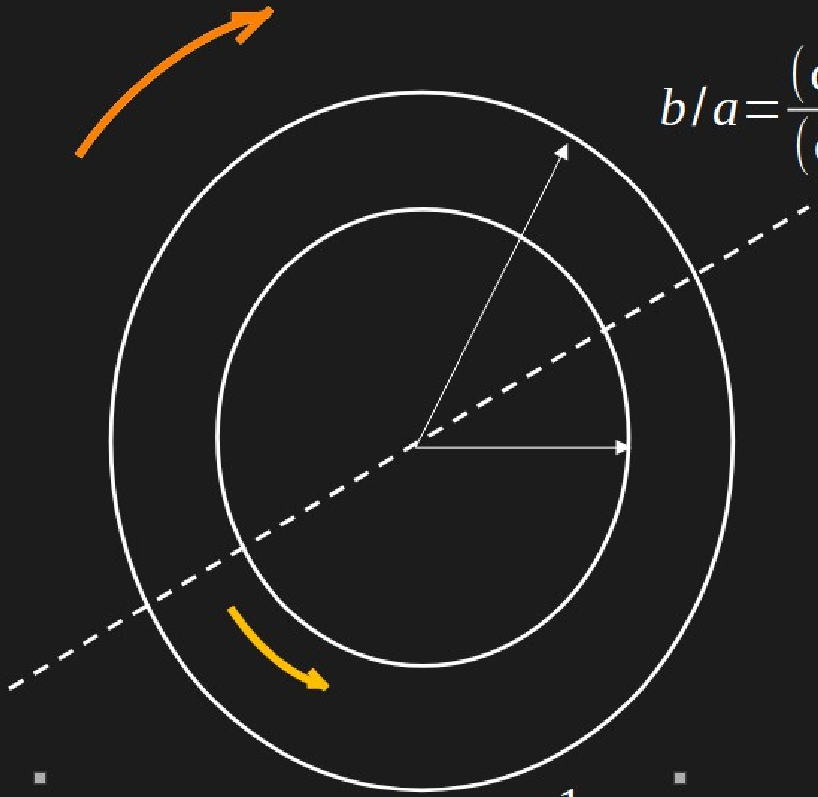
$$\theta,\varphi \rightarrow z=\cot(\theta/2)\exp(i\varphi)$$

Two vectors are 'equivalent' i.e represent the same state of polarisation, if they only differ by an overall complex multiplication

The ratio of the two complex numbers encodes the relative amplitudes and phases

$$\begin{bmatrix}\cos(\theta/2)\exp(i\phi)\\ \sin(\theta/2)\end{bmatrix}:=\begin{bmatrix}z_L\\ z_R\end{bmatrix}\equiv\begin{bmatrix}z=z_L/z_R\\ 1\end{bmatrix}$$

Using the circular basis



$$b/a = \frac{(\cos(\theta/2) - \sin(\theta/2))}{(\cos(\theta/2) + \sin(\theta/2))}$$

$$\tan((\pi/2 - \theta)/2) = \tan\left(\frac{1}{2} \quad \text{latitude on the Poincare sphere}\right)$$

Intensity and inteference

- The cycle averaged intensity (flux, power) is given by

$$\langle \vec{E}\cdot\vec{E}\rangle = \frac{1}{2}\left(a_1^2 + a_2^2 = E_x^* E_x + E_y^* E_y = \begin{bmatrix} E_x^* & E_y^* \end{bmatrix}\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \boldsymbol{E}^+ \boldsymbol{E}\right)$$

- $\boldsymbol{E} = \boldsymbol{E}_a + \boldsymbol{E}_b$ superposing two waves of different polarisations

- In terms of the two pieces of ***E*** this reads

$$\boldsymbol{I} = \boldsymbol{E}_a^{+}\boldsymbol{E}_a + \boldsymbol{E}_b^{+}\boldsymbol{E}_b + \boldsymbol{E}_a^{+}\boldsymbol{E}_b + \boldsymbol{E}_b^{+}\boldsymbol{E}_a$$

Dot/ inner product, orthogonality

- We have the sum of individual intensities plus an interference term plus its complex conjugate which is the dot / inner product for complex vectors
- Looks as if we have only the real part of $E_a^+ E_b$. But the imaginary part is also accessible by inserting an extra 90 degrees of phase.
- When this term is zero we do not see any fluctuations in intensity as the relative phase is varied - such states are called *orthogonal*
- Examples, two perpendicular linearly polarised waves, two opposite circular waves (Fresnel, Arago)
- Most general case, two ellipses, same axial ratio, perpendicular major axes, opposite senses of traversal

Checking orthogonality of antipodes on the PS

$$\boldsymbol{E_a} = \begin{bmatrix} \cos(\theta/2)\exp(i\phi) \\ \sin(\theta/2) \end{bmatrix} \text{ and } \boldsymbol{E_b} = \begin{bmatrix} -\sin(\theta/2)\exp(i\phi) \\ \cos(\theta/2) \end{bmatrix}$$

Easily verified that $\boldsymbol{E}_a^+ \boldsymbol{E_b} = 0$

“Gains”

- Multiplication of the vector by a complex gain ***g*** is an overall amplification/ attenuation and common phase.
- The most general 2x2 matrix ***G*** acting on the left gives a column vector with two numbers. This describes a general mixing / amplification / attenuation of the individual field components, as in a non-ideal measurement
- ***G*** is usually split into complex gains for each polarisation and leakage (D) terms
- ***g, G*** need to be calibrated to convert the measured correlations into those of the incoming electric field

A dictionary between radio telescope polarisation and linear algebra

- Incoming electric field
- Power / flux
- feed
- Response of feed to field
- Quadrature hybrid
- Lossless device
- Lossy leaky device
- WATCH THIS SPACE

- Complex column vector
- Norm
- Row vector (dual space)
- Inner product (Row times column)
- Change (to a circular) basis
- Unitary transformation
- General linear transformation
- Outer product (column x row)

What good is the Poincare sphere?

- Gets the topology of polarisation right
- Opposite points are orthogonal states
- States separated by γ overlap by a factor $\cos(\gamma/2)$
- Transformations which leave two opposite states unchanged are rotations around the axis joining those two points (these are the lossless, ‘unitary’ transformations)
- Any basis change simply rotates the sphere about some axis
- More to come in Lecture 3